Drifting Markov models applied in reliability theory and survival analysis

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Résumé. Dans ce travail nous nous intéressons aux systèmes multi-états modélisés par une classes particulière de processus stochastiques, appelés processus de Markov dérivants ; nous étudions des mesures associées de fiabilité/analyse de survie et nous estimons ces quantités pour des données statistiques variées.

Des outils classiques pour étudier l'évolution et la performance d'un tel système sont les processus markoviens et semi-markoviens (cf. Sadek et Limnios, 2002 ; Limnios et Ouhbi, 2006 ; Barbu et Limnios, 2008). Une hypothèse trop souvent utilisée est celle d'homogénéité par rapport au temps ; il est clair que cette homogénéité n'est pas adaptée dans beaucoup d'applications. Mais, d'un point de vue pratique, considérer des processus non-homogènes peut être inapproprié. Une solution possible est de considérer une non-homogénéité "lisse", contrôlée, d'une forme connue. Un exemple de ce type dans un contexte markovien est fournit par ces processus de Markov dérivants (cf. Vergne 2008). Pour ces processus, la matrice de transition est une fonction linéaire (polynomiale) de deux (plusieurs) matrices de transition markoviennes. Nous obtenons ainsi la non-homogénéité "lisse" désirée.

Mots-clés. Chaînes de Markov dérivantes, fiabilité, analyse de survie, propriétés asymptotiques

Abstract. In this work we focus on multi state systems modelled by means of a particular class of stochastic processes called drifting Markov processes ; we investigate associated reliability/survival indicators and estimate these quantities under various statistical shemes.

Typical tools for studying the evolution and performance of such systems are the Markov and semi-Markov processes (cf. Sadek and Limnios, 2002 ; Limnios and Ouhbi, 2006 ; Barbu and Limnios, 2008). A hypothesis used in many mathematical models built as modelling tools for real applications is the homogeneity with respect to time ; clearly, in many applications this homogeneity is inappropriate. But, from a practical point of view, considering general non-homogeneous processes could be inappropriate. A possible solution is to consider a non-homogeneity that is "smooth", controlled, of a known shape. An example of this type in a Markov framework consists in these drifting Markov chains

(cf. Vergne 2008). For these processes, the Markov transition matrix is a linear (polynomial) function of two (several) Markov transition matrices. Thus we obtain the desired "smooth" non-homogeneity.

Keywords. Drifting Markov chains, reliability theory, survival analysis, estimation, asymptotic properties

1 Drifting Markov models

Let us consider a random system with finite state space $E = \{1, \ldots, s\}$, $s < \infty$, and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. All along this paper we assume that the time evolution of the system is governed by discrete-time stochastic processes defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in E. Let $\mathbf{\Pi}_0 = (\Pi_0(u,v))_{u,v \in E}$ and $\mathbf{\Pi}_1 = (\Pi_1(u,v))_{u,v \in E}$ be two Markov transition matrices (of order 1) over a state space E.

Definition 1 (linear drifting Markov chain of order 1 and of length n)**.** *A sequence $X_0, X_1, \ldots, X_n$ with state space $E = \{1, 2, \ldots, s\}$ is said to be a* linear drifting Markov chain (of order 1) *of length n between the Markov transition matrices $\mathbf{\Pi}_0$ and $\mathbf{\Pi}_1$ if the distribution of X_t, $t = 1, \ldots, n$, is defined by*

$$\mathbb{P}(X_t = v \mid X_{t-1} = u, X_{t-2}, \ldots) = \Pi_{\frac{t}{n}}(u,v), \; u, v \in E, \tag{1}$$

where

$$\Pi_{\frac{t}{n}}(u,v) = \left(1 - \frac{t}{n}\right) \Pi_0(u,v) + \frac{t}{n} \Pi_1(u,v), \; u, v \in E. \tag{2}$$

Let us denote by $\boldsymbol{\alpha} = (\alpha(1), \ldots, \alpha(s))$ the *initial distribution of the chain*, that is the distribution of X_0, $\alpha(u) = \mathbb{P}(X_0 = u)$ for any state $u \in E$.

Note that one could consider a k order linear drifting Markov chain, $k \in \mathbb{N}^*$, by allowing a dependence of order k in Equation (1) and letting $\mathbf{\Pi}_0$ et $\mathbf{\Pi}_1$ be the transition probabilities of Markov chains of order k. Note also that the linear drifting can be generalized to polynomial drifting of a certain degree d. See Vergne (2008) for more details on these points. Let us generalize *linear drifting Markov model* of order 1 to *polynomial drifting Markov model* of order k and degree d. Let $\mathbf{\Pi}_{\frac{i}{d}} = (\mathbf{\Pi}_{\frac{i}{d}}(u_1, \ldots, u_k, v))_{u_1, \ldots, u_k, v \in E}$ be d Markov transition matrices (of order k) over a state space E.

Definition 2 (polynomial drifting Markov chain of order k and of length n)**.** *A sequence $X_0, X_1, \ldots, X_n$ with state space $E = \{1, 2, \ldots, s\}$ is said to be a* polynomial drifting Markov chain of order k *and of length n if the distribution of X_t, $t = 1, \ldots, n$, is defined by*

$$\mathbb{P}(X_t = v \mid X_{t-1} = u_k, X_{t-2} = u_{k-1}, \ldots) = \mathbf{\Pi}_{\frac{t}{n}}(u_1, \ldots, u_k, v), \; u_1, \ldots, u_k, v \in E, \tag{3}$$

where

$$\mathbf{\Pi}_{\frac{t}{n}}(u_1,\ldots,u_k,v) = \sum_{i=0}^{d} A_i(t)\mathbf{\Pi}_{\frac{i}{d}}(u_1,\ldots,u_k,v),\ u_1,\ldots,u_k,v\in E. \quad (4)$$

with A_i polynomials of degree d such as, for any $i,j\in\{0,1,\ldots,d\}$,

$$A_i\left(\frac{nj}{d}\right) = \mathbb{1}_{\{i=j\}}.$$

2 Reliability and survival measures

In order to undertake a reliability analysis of a system modelled by a drifting Makov chain, let us assume that the state space of the system is partitioned into working and failure states, $E = U \cup D$, with $U \cap D = \emptyset$, where $U = \{1,\ldots,s_1\}$ represents the working states and $D = \{s_1+1,\ldots,s\}$ the failure states of the system. According to this partition of the state space we partition any matrix of vector we are working with. For instance, we have

$$\mathbf{\Pi}_0 = \begin{array}{c} \begin{matrix} U & D \end{matrix} \\ \begin{pmatrix} \mathbf{\Pi}_0^{UU} & \mathbf{\Pi}_\mathbf{0}^{UD} \\ \mathbf{\Pi}_0^{DU} & \mathbf{\Pi}_\mathbf{0}^{DD} \end{pmatrix} \end{array} \begin{matrix} U \\ D \end{matrix}, \qquad \boldsymbol{\alpha} = \begin{array}{c} \begin{matrix} U & D \end{matrix} \\ \begin{pmatrix} \boldsymbol{\alpha}^U & \boldsymbol{\alpha}^D \end{pmatrix} \end{array}.$$

For a system starting to function at time $l = 0$, the *reliability* at time $l \in \mathbb{N}$ is the probability that the system has functioned without failure in the period $[0,l]$. Denoting by T_D hitting time of subset D, called the *lifetime of the system*, i.e.,

$$T_D := \inf\{t\in\mathbb{N};\quad X_t \in D\},\ \text{with}\ \inf\emptyset := \infty,$$

the reliability at time $l \in \mathbb{N}$ is

$$R(l) := \mathbb{P}(T_D > l) = \mathbb{P}(X_t \in U, t = 0,\ldots,l).$$

The following results will present the expressions of reliability measures.

Proposition 1. *For a linear drifting Markov chain of order* 1 $(X_t)_{0\le t\le n}$, *the reliability at time l, $l \in \mathbb{N}$, is given by*

$$R(l) = \boldsymbol{\alpha}^U \prod_{t=1}^{l}\left(\left(1-\frac{t}{n}\right)\mathbf{\Pi}_0^{UU} + \frac{t}{n}\mathbf{\Pi}_1^{UU}\right)\mathbb{1}^U, \quad (5)$$

where $\mathbb{1}^U = (\underbrace{1,\cdots,1}_{s_1})^\top$.

Proposition 2. *For a linear drifting Markov chain of order* 1 $(X_t)_{0\leq t\leq n}$, *the pointwise (or instantaneous) availability at time* l, $l \in \mathbb{N}$, *is given by*

$$A(l) = \boldsymbol{\alpha} \prod_{t=1}^{l} \left(\left(1 - \frac{t}{n}\right) \mathbf{\Pi}_0 + \frac{t}{n} \mathbf{\Pi}_1 \right) \ \mathbb{1}^{E,U}, \tag{6}$$

where $\mathbb{1}^{E,U} = (\underbrace{1, \cdots, 1}_{s_1}, \underbrace{0, \cdots, 0}_{s-s_1})^\top$.

Let us assume that we deal with a repairable system for which the failure occurs at time $l = 0$; its *maintainability* at time $k \in \mathbb{N}$ is the probability that the system is repaired up to time l, given that it has failed at time $l = 0$.

In other words, we take $\boldsymbol{\alpha}^U = \mathbf{0}$ and we denote by T_U the first hitting time of subset U, called the *duration of repair* or *repair time*, that is,

$$T_U = \inf\{t \in \mathbb{N};\ X_t \in U\}.$$

Thus, the maintainability at time $l \in \mathbb{N}$ is defined by

$$M(l) \ := \ \mathbb{P}(T_U \leq l) = 1 - \mathbb{P}(T_U > l) = 1 - \mathbb{P}(X_t \in D, t = 0, \ldots, l).$$

Proposition 3. *For a linear drifting Markov chain of order* 1 $(X_t)_{0\leq t\leq n}$, *the maintainability at time* l, $l \in \mathbb{N}$, *is given by*

$$M(l) = 1 - \boldsymbol{\alpha}^D \prod_{t=1}^{l} \left(\left(1 - \frac{t}{n}\right) \mathbf{\Pi}_0^{DD} + \frac{t}{n} \mathbf{\Pi}_1^{DD} \right) \ \mathbb{1}^D, \tag{7}$$

where $\mathbb{1}^D = (\underbrace{1, \cdots, 1}_{s-s_1})^\top$.

3 Estimation of drifting Markov models

The purpose of this section is twofold : first, continuing the direction developed in Vergne (2008), we will consider different types of data for which the estimators of the characteristics of a drifting Markov chain will be derived. Second, we will estimate the associated reliability indicators.

Let us now consider H i.i.d. sample paths of a linear drifting Markov chain $(X_k)_{0\leq k\leq n}$,
— $\mathcal{H}_1(n) := (X_0^1, X_1^1, \ldots, X_n^1)$,
— $\mathcal{H}_2(n) := (X_0^2, X_1^2, \ldots, X_n^2)$,
— $\vdots$
— $\mathcal{H}_H(n) := (X_0^H, X_1^H, \ldots, X_n^H)$.

That is, we assume that the sample paths have the same length n and that the linear drift within the H sample paths takes place between the same two Markov transition matrices (of order 1) Π_0 and Π_1.

Proposition 4. *Under this setting, starting with H sample paths of a linear drifting Markov chain between two Markov transition matrices (of order 1) $\mathbf{\Pi}_0$ and $\mathbf{\Pi}_1$, for any states $u, v \in E$, the estimators of $\Pi_0(u,v)$ and $\Pi_1(u,v)$ are given by :*

$$\widehat{\Pi}_{0;(n,H)}(u,v) = \frac{\left(\begin{array}{c} \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(\frac{t}{n}\right)^2\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u, X_t^h=v\}}\right)\left(1-\frac{t}{n}\right)\right) - \\ \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)\left(\frac{t}{n}\right)\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u, X_t^h=v\}}\right)\left(\frac{t}{n}\right)\right) \end{array}\right)}{\left(\begin{array}{c} \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)^2\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(\frac{t}{n}\right)^2\right) - \\ \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)\left(\frac{t}{n}\right)\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)\left(\frac{t}{n}\right)\right) \end{array}\right)},$$

$$\widehat{\Pi}_{1;(n,H)}(u,v) = \frac{\left(\begin{array}{c} \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)^2\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u, X_t^h=v\}}\right)\left(\frac{t}{n}\right)\right) - \\ \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)\left(\frac{t}{n}\right)\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u, X_t^h=v\}}\right)\left(1-\frac{t}{n}\right)\right) \end{array}\right)}{\left(\begin{array}{c} \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)^2\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(\frac{t}{n}\right)^2\right) - \\ \left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)\left(\frac{t}{n}\right)\right)\left(\sum_{t=1}^{n}\left(\sum_{h=1}^{H}\mathbb{1}_{\{X_{t-1}^h=u\}}\right)\left(1-\frac{t}{n}\right)\left(\frac{t}{n}\right)\right) \end{array}\right)}.$$

Note that we consider various types of data that can occur in practice and adapt the estimation procedures that we have previously obtained in order to get estimators of the drifting Markov models in these cases. These different types of data arise from three ingredients : an observed sample path varying between two Markov transition matrices could be complete or incomplete ; we can have at our disposal either one or several sample paths ; several observed sample paths could be of the same length or of different length.

Using the expression of the reliability indicators of a drifting Markov chain previously obtained and the estimators of the characteristics of a drifting Markov chain, one immediately obtains the associated plug-in estimators of the reliability metrics, $\widehat{R}_{(n,H)}(l)$, $\widehat{A}_{(n,H)}(l)$, $\widehat{M}_{(n,H)}$.

Theorem 1. *For a linear drifting Markov chain $(X_k)_{0\leq k\leq n}$ between two Markov transition matrices (of order 1) $\mathbf{\Pi}_0$ and $\mathbf{\Pi}_1$, for any states $u, v \in E$ and time $t, t = 1, \ldots, n$, the obtained estimators are strongly consistent, i.e.,*

$$\widehat{\Pi}_{0;(n,H)}(u,v) \xrightarrow[H\to\infty]{a.s.} \Pi_0(u,v)$$
$$\widehat{\Pi}_{1;(n,H)}(u,v) \xrightarrow[H\to\infty]{a.s.} \Pi_1(u,v)$$
$$\widehat{\Pi}_{\frac{t}{n};H}(u,v) \xrightarrow[H\to\infty]{a.s.} \Pi_{\frac{t}{n}}(u,v)$$
$$\widehat{R}_{(n,H)}(l) \xrightarrow[H\to\infty]{a.s.} R(l);$$
$$\widehat{A}_{(n,H)}(l) \xrightarrow[H\to\infty]{a.s.} A(l);$$
$$\widehat{M}_{(n,H)}(l) \xrightarrow[H\to\infty]{a.s.} M(l).$$

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