A concentration inequality for a Gaussian process indexed by matrices

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Résumé. On obtient une inégalité de concentration pour un type spécifique de l'intégrale de Itô indexée par matrices. Ce type de processus stochastiques apparat dans le cadre de l'estimation d'une fonction multivariée non paramétrique.

Mots-clés. Inégalité de concentration, intégrale de Itô, estimation d'une fonction non paramétrique.

Abstract. A concentration inequality for a specific type of the Itô integrals indexed by matrices is obtained. This type of stochastic processes appears in the framework of nonparametric multivariate function estimation.

Keywords. Concentration inequality, Itô integral, nonparametric function estimation.

1 Introduction

The concentration-of-measure is one of the classical problems and has been widely studied over the last four decades. It has recently become one of the most powerful tools in nonparametric statistics, particularly in machine learning and adaptive estimation, see Boucheron, Lugosi and Massart (2013) and the references therein for an overview of recent advances in the field.

It is worth mentioning that the concentration inequalities for stochastic processes indexed by matrices are not well studied yet. The purpose of this note is to establish a result that appears in the context of structural adaptive methods like the ones of Goldenshluger and Lepski (2008), Goldenshluger and Lepski (2009), Lepski and Serdyukova (2013) and Lepski and Serdyukova (2014). In statistics, the single-index modeling, $F(\theta) = f(x^\top\theta)$, being a natural relaxation of the generalized linear models implies that both the link function f and the index vector θ are unknown and therefore must be estimated. In this case, the entries of the matrix E, see (1), are the components of the vector θ and some regularization parameters related to nonparametric estimation methods.

2 Concentration inequality

Let $\mathcal{D}$ be a bounded interval in $\mathbb{R}^d$ and $\{W(t), t \in \mathcal{D}\}$ a Brownian sheet. Let an index set $\mathcal{E}_{a,A},\ 0 < a, A < \infty$, be a set of $d \times d$ matrices such that

$$|\det(E)| \geq a, \quad |E|_\infty \leq A,\ \ \forall E \in \mathcal{E}_{a,A}.$$

Here, $|E|_\infty = \max_{i,j} |E_{i,j}|$ denotes the matrix supremum norm, the maximum absolute value entry of the matrix E.

Assume that the function $\mathcal{L} : \mathbb{R}^d \to \mathbb{R}$ is compactly supported on $[-1/2, 1/2]^d$, integrates to one, $\int \mathcal{L} = 1$, and satisfies the Lipschitz condition

$$|\mathcal{L}(u) - \mathcal{L}(v)| \leq \Upsilon |u - v|_2,\ \ \forall u, v \in \mathbb{R}^d,$$

where $|\cdot|_2$ is the Euclidian norm. Let $y \in \mathbb{R}^d$ be fixed. On the parameter set $\mathcal{E}_{a,A}$, let a Gaussian random function be defined by

$$\zeta_y(E) = \|\mathcal{L}\|_2^{-1} \sqrt{|\det(E)|} \int \mathcal{L}\big(E(u-y)\big) W(\mathrm{d}u), \tag{1}$$

where $\|\cdot\|$ is a standard L_2 norm. Put $c(d) = 2\sqrt{10 \ln d + 2(d-1)\ln(d-1)}$ and

$$\mathbf{c}_d(a, A) = d \left\{ 4\sqrt{\ln\Big(\Big[(A^d/a)^{1/2} \vee (A^d/a)\Big]\Big)} + c(d) + 4\sqrt{\ln(1+\Upsilon)} + 13 \right\}.$$

Proposition 1 *For any $z > 0$*

$$\mathbb{P}\left\{ \sup_{E \in \mathcal{E}_{a,A}} |\zeta_y(E)| \geq \mathbf{c}_d(a, A) + z \right\} \leq 2e^{-\frac{z^2}{2}}.$$

Moreover, for any $q \geq 1$

$$\left(\mathbb{E}\Big[\sup_{E \in \mathcal{E}_{a,A}} \big|\zeta_y(E)\big| \Big]^q \right)^{1/q} \leq \big(1 + (2\pi)^{1/(2q)} \mathfrak{c}_q\big)\, \mathbf{c}(a, A)$$

with $\mathfrak{c}_q = \big(\mathbb{E}\big(1 + |\varsigma|\big)^q\big)^{1/q}$, where $\varsigma \sim \mathcal{N}(0, 1)$.

This result generalizes Lepski and Serdyukova (2013).

Bibliographie

[1] Boucheron, S., Lugosi, G., Massart, P. (2013), *Concentration inequalities*, Oxford university press, Oxford.

[2] Goldenshluger, A., Lepski, O. (2008), *Universal pointwise selection rule in multivariate function estimation*, Bernoulli 14. 11501190.
[3] Goldenshluger, A., Lepski, O. (2009), *Structural adaptation via L_p-norm oracle inequalities*, Probab. Theory Related Fields 143, 41–71.
[4] Lepski O., Serdyukova N. (2013), *Adaptive estimation in the single-index model via oracle approach*, Math. Meth. Statist. 22, 310–332.
[5] Lepski O., Serdyukova N. (2014), *Adaptive estimation under single-index constraint in a regression model*, Ann. Statist. 42, 1–28.